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Earth tides, ocean tides and tidal loading

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The distortion of the Earth's gravitational potential field by the M_2 ocean tide has been calculated, accounting for both the elastic deformation of the Earth and the self potential of the water. The potential field generated by the ocean tide is almost everywhere greater than a tenth, and over much of the ocean is half as large as the lunar driving potential itself, and may have a significant influence on the tidal motion. Load tides in tilt, strain, and vertical acceleration also arise from the deformation of the Earth by the ocean tide. These load tides are probably of more geophysical and oceanographic interest than the body tides raised by the Sun and Moon.

Introduction

Earth tides are interesting because of ocean tidal loading. If the Earth had no ocean, or if the tide in the ocean was in equilibrium with the tide potential, then a few accurate measurements of the solid Earth tide would exhaust most of the interest in this classical geophysical discipline. A small number of observations would supply the handful of constants necessary to describe the Earth and ocean tides everywhere on such an ideal world. (There is a tacit assumption here that the Earth's lateral heterogeneity has only a small effect on the tides.)

Earth tides on the real world are more complicated, however, because they are significantly perturbed by the loading of the highly irregular tides in the world's oceans. On the other hand, the ocean tide is itself affected by the deformation of the sea floor under the weight of the tidal water. This deformation can create forces, arising from the perturbation of the Earth's gravity field, which are comparable to the original astronomical driving force. Because of the intimate relation between Earth tides and ocean tides, neither subject can be completely understood without consideration of the other, and we are just now beginning this geophysical-oceanographic collaboration.

A tide is a complicated phenomenon, and the first step in any investigation of the tides is to resolve the time dependence into its constituent frequencies by harmonic analysis. The various frequencies are related to a few basic periodicities in the motion of the Earth, Moon and Sun, and harmonic analysis of the tides is useful because the ocean tide, the equilibrium earth tide and the load tide are all linear functions of the astronomical driving force. There are, in fact, some areas, such as the Gulf of Alaska, where the ocean tide (and presumably the load tide) is somewhat nonlinear, but these regions are few and can be ignored for the present. Unless it is stated otherwise, we assume such a harmonic expansion and most of the following discussion applies in particular to the largest tidal constituent, the semidiurnal lunar part with period 12.42 h (M_2). Furthermore, all functions are complex, since they have both amplitude and phase.

2. Ocean tides on an elastic Earth

In every important harbour the tide is accurately known and can be precisely predicted years in advance. In the deep sea, however, the tide is almost unknown, with the few exceptions

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where it has been measured with ocean bottom recorders. There is virtually no similarity between the tides on opposite shores of an ocean basin and in general maps showing amplitudes and phases of deep sea tides are very uncertain. The tide is spatially irregular because the oceans resonate to forces of tidal periodicity, and the large scale-length driving force excites small scale-length oscillations around a dozen nodal points, called amphidromes.

Until the ocean tide has been mapped experimentally, it is necessary to adopt an ocean tide model in order to study the Earth tide–ocean tide interaction. I am working with numerical models calculated by Hendershott (1972) who solves the Laplace tidal equations (L.t.e.) subject to various boundary conditions with a finite difference scheme. The usual boundary condition is that the calculated tide matches the observed tide along all coastlines, but this has the severe limitation that there are few tide measurements on one of the most important boundaries, the coast of Antarctica. Hendershott assumes the tide is nondissipative. Other calculations have been presented by Pekeris & Accad (1970), Bogdanov & Magarik (1967), and Tiron, Sergeev & Michurin (1967). A review of empirical and numerical models of the ocean tides has recently been given by Hendershott & Munk (1970).

(*a*) *The Laplace tidal equations*

The Laplace tidal equations, expressing the conservation of mass and momentum, are not separable because of the Coriolis force associated with the Earth's rotation. They can be combined into the single equation

$$L(Z) = \mathrm{i}\sigma\,[\alpha(U_2/g)+(P/g)], \tag{1}$$

where $L(\ )$ is an elliptic, partial differential operator, and σ the radian frequency (Hendershott 1972). The dependent variable is

$$Z = H-\alpha(U_2/g)-(P/g), \tag{2}$$

with H the water amplitude measured with respect of the deformed sea floor. Along coast lines Z is fixed because H must be the observed tidal amplitude. In equations (1) and (2), U_2 is the potential of the astronomical driving force (at latitude θ and east longtidue ϕ, $U_2/g = 0.247 \cos^2\theta \cos(\sigma t+2\phi)$ m for the lunar M_2 tide) and

$$\alpha = 1+k_2-h_2, \tag{3}$$

where $k_2 \approx 0.3$ and $h_2 \approx 0.6$ are the dimensionless Love numbers describing the Earth's elastic yielding to the equilibrium potential U_2. P is the self potential of the tidal water plus the potential field created by the Earth's elastic deformation under the weight of the tide. By fixing $k_2 = h_2 = P \equiv 0$, equation (1) applies to the tides of a massless ocean on a rigid Earth. Note that the Earth's elastic yielding enters into (1) in two distinct ways: the equilibrium Earth tide reduces the effective astronomical potential by 30%; and the elastic loading of the sea floor enters into P.

More or less realistic solutions to (1) can be readily obtained for oceans of irregular outline and variable depth when $P = 0$. It is more difficult to account for self attraction and loading, however, because P is an integral function of the unknown solution H. Let $\boldsymbol{r}$, $\boldsymbol{r}'$ be position vectors on the surface of the Earth, ρ the density of sea water and G a gravitational potential Green function. Then P can be written as the convolution integral

$$P(\boldsymbol{r}) = \rho\iint_{\text{oceans}} G(|\boldsymbol{r}-\boldsymbol{r}'|)H(\boldsymbol{r}')\,\mathrm{d}A. \tag{4}$$

The Green function G can be calculated for any radially stratified Earth model by solving the elastic and Poisson equations, with the boundary condition that a unit mass press on the Earth's free surface (Farrell 1972*b*). In terms of the *load* Love numbers k_n and h_n,

$$G(\theta) = \frac{ag}{m_{\mathrm{E}}} \sum_{n=0}^{\infty} (1+k_n-h_n) P_n(\cos\theta), \tag{5}$$

with a, g, and m_{E} the Earth's radius, surface gravity and total mass.

Given the Green function G, equation (1) is solved iteratively. Define a sequence of functions $(H^0, H^1, \ldots)$, and $(P^0, P^1, \ldots)$ with $P^0 = 0$ and for $n \geqslant 1$

$$P^n = G * H^{n-1}, \tag{6}$$

where $*$ is the convolution operator. For each P^n, equation (1) is solved to yield Z^n and hence by (2) H^n. From H^n, P^{n+1} is found and the process repeated until two successive solutions are similar enough. We hope this method converges.

(*b*) *A preliminary convolution*

One of Hendershott's first-order solutions of the L.t.e., computed on a 6° Mercator grid, with $P = 0$, was used to find P^1. The gravitational potential perturbation caused by the tide (plotted in Hendershott & Munk 1970) is shown in figure 1, where the Green function for the

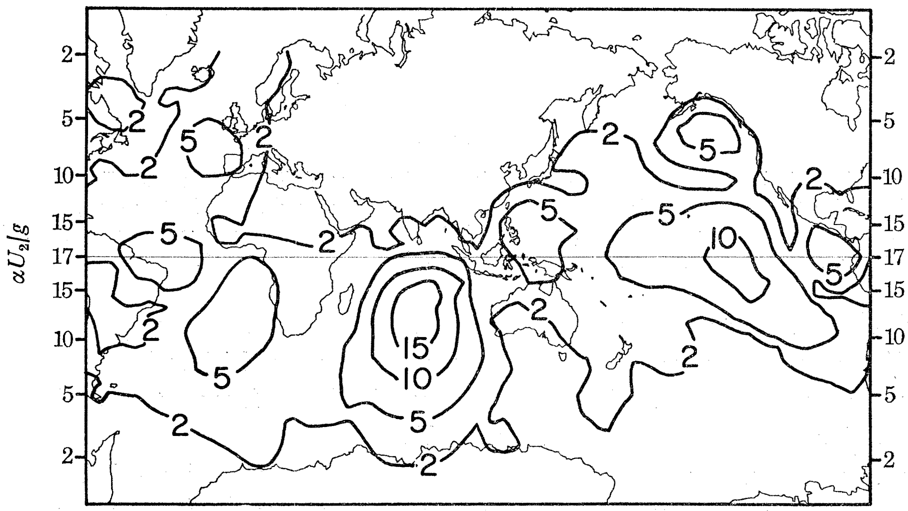

FIGURE 1. The magnitude of P^1/g for one M_2 ocean tide model. Values of the magnitude of $\alpha U_2/g$, which is constant along lines of latitude, are shown in the margins for comparison. Contour levels in centimetres. The Indian ocean high is unrealistically large by a factor of 2 and is caused by the large tidal amplitudes there in the ocean tide model. The phase angles of P^1/g and $\alpha U_2/g$ also depend on location, but lines of constant phase are not shown here.

Gutenberg–Bullen Earth with an oceanic upper mantle has been used in the convolution. Details of the calculation of the various Green functions for evaluating the response of earth models to surface mass loads were reviewed by Farrell (1972*b*), but that article was concerned with the displacements, accelerations, tilts, and strains, not with the gravitational potential perturbation.

A study of the vertical acceleration due to tidal loading (Farrell 1972*a*), showed that it was necessary for the tide to conserve mass. The original tide model did not, but it can be adjusted so there is no flow of water across the ocean boundaries by subtracting 6.8 exp$(-i56)$ cm of water from the tide. An alternative method of achieving the same result is to reject the Legendre harmonic of degree 0 from the convolution. This is easily done by modifying the Green function so that the sum in equation (5) starts at $n = 1$ instead of $n = 0$. The convolution has been done both ways and P^1 evaluated with the modified tide is very similar to P^1 evaluated with the modified Green function. The slight differences are due to the $n \geqslant 1$ harmonics of the ocean function (Munk & MacDonald 1960, p. 289) which contribute to the modified tide integral but not to the modified Green function integral. The modified Green function was used to calculate figure 1.

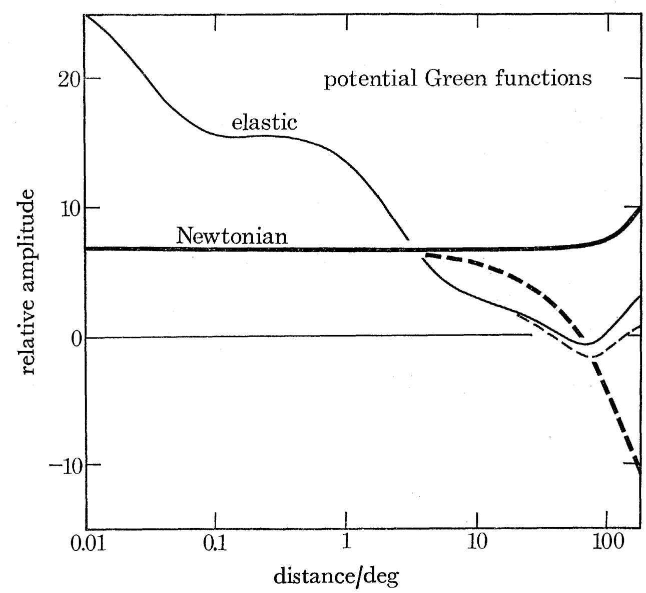

FIGURE 2. The elastic and Newtonian parts of the gravitational potential Green function. The amplitudes have been multiplied by $(a\theta/g) \times 10^{12}$ (a is 6.371×10^6 m, θ is distance in radians, g is $9.8\ \mathrm{m\,s^{-2}}$) to give relative amplitudes. Omitting the Legendre degree 0 harmonic from the sum gives the two dashed curves. The sum of the two dashed curves is the modified Green function used to produce figure 1.

The surprising result from these calculations is the magnitude of P^1/g compared to $\alpha U_2/g$. In the oceans P^1/g is almost always greater than a tenth $\alpha U_2/g$, and at high latitudes and where the ocean tide is large or of constant phase it can equal the lunar forcing term. We do not yet know the effect this will have on the iteration for H^1.

A rapid scheme is necessary to calculate the convolution integral. P is needed at about 2000 points on this 6° grid, so each two-dimensional convolution integral must be found quickly. Use of Simpson's or some other integration rule for each point would be a lengthy procedure. Instead, G is integrated once across disks located at several distances r, and from these few integrations a table of disk factors is constructed. The entries in this table are corrections which must be applied to the Green function when the load originates in a disk of radius α whose centre is distance r away. Around each ocean grid point, then, a disk is constructed within which the tide is assumed to be uniform. The convolution integral is now found by interpolating tabular values of the Green function and disk factors. When $\alpha/r < 0.1$, it is sufficient to assume the finite load originates at a point, and the geometric correction can be dispensed with.

The Green function, suitably normalized, is plotted in figure 2. To exhibit the relative importance of the self attraction of the water (the Newtonian part of G) versus the yielding of the Earth (the elastic part of G), the two terms are plotted separately in the figure. The elastic and

Newtonian effects are seen to be equal at a distance of about 3°. Closer than this to the load, the elastic term dominates G, and at further distances G is mostly the self attraction of the water. When G is convolved with the tide, about half the potential perturbation P is due to the nearby ($< 3°$) elastic deformation and about half arises from the integrated Newtonian effect. The elastic Green function itself is partly due to the perturbation in the Earth's gravity field, the k_n sum in equation (5), and partly due to the vertical deformation of the surface, the h_n sum. The latter is the most important, and nowhere does the k_n term contribute more than 10 %

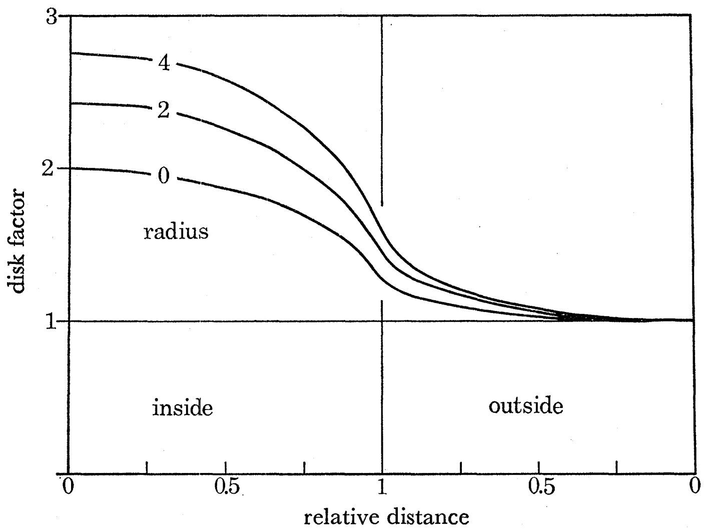

FIGURE 3. Disk factors obtained by integrating the gravitational potential Green function across disks of several radii α (degrees) centred at various distances r. When $r < \alpha$, the inside case, the relative distance is r/α, and the disk factor is the integral divided by $G(\alpha)$. When $r > \alpha$, the outside case, the relative distance is α/r and the integral is divided by $G(r)$ to obtain the disk factor. The curve labelled 0° applies to $\alpha \leqslant 10^{-5}$ degrees, and is the half space limit.

to the elastic Green function. The dashed curves in figure 1 show the modified Green functions, obtained by subtracting $agP_0(\cos\theta)/m_{\mathrm{E}}$ and $-agh_0\, P_0(\cos\theta)/m_{\mathrm{E}}$ ($k_0 = 0$, $h_0 = -0.134$) from the two responses.

The Green function G is integrated across circular disks of varying radii α to get the disk factors plotted in figure 3. When $r > \alpha$ the integral is divided by $G(r)$ to get the correction, but when $r < \alpha$, the integral is normalized by $G(\alpha)$. The curve for $\alpha = 0$ is the relative response for a disk load on a half-space, and is the limit towards which the other curves tend.

3. Earth tides and tidal loading

The tilt, strain and gravity changes arising from the deformation of the Earth by ocean tide loads are easily measured with modern geophysical instruments. This is particularly true along coastlines and on ocean islands, but even in the interior of a continent the ocean tide can cause a 2 % perturbation in the gravity tide (Kuo, Jachens, Ewing & White 1970). The geophysical effects of any ocean tide model are found by evaluating the convolution integral (equation (4)), where G is the appropriate Green function for some Earth model. Farrell (1972*b*) has tabulated for three gravitating and radially stratified Earth models the Green functions for displacement, tilt, vertical acceleration and strain.

Numerous quantitative studies of this type have shown how well ocean loading accounts for most of the irregularities in the Earth tides (Farrell 1970; Kuo *et al.* 1970; Prothero & Goodkind 1972; Beaumont & Lambert 1972). Near oceans the load perturbation can reach 10% of the equilibrium gravity tide, but for strain and tilt the load tide may equal or vastly exceed the equilibrium tide. On the other hand, the tilt and strain perturbations decrease more rapidly inland than the gravity perturbation because the r^{-2} tilt and strain Green functions are more localized than the r^{-1} gravity Green function.

To study the large coastal loads, which are generated predominately by the tide immediately offshore, the adjacent ocean tide must be modelled on a denser grid than is required for global studies. The best procedure is probably to patch offshore tide models (Munk, Snodgrass & Wimbush 1970) into the global model, when near shore data are available. It is also desirable to use the Green functions appropriate to the local Earth structure. The far field load response is insensitive to the near surface properties of the Earth, so the Green function for the Gutenberg–Bullen or some other standard earth model can be taken at distances beyond 10°, regardless of the local Earth structure, and only the near field response needs to be recalculated. This can be adequately done ignoring the Earth's sphericity and the self gravitational forces.

4. Future prospects

The traditional use of Earth tides has been to study the Earth's deformation under the lunar and solar body forces, and from the observed deformation to infer the Love numbers h_2, k_2, l_2. Since h_2, k_2, l_2 depended on Earth structure, precise experimentally observed values for these constants would limit the class of acceptable Earth models. But the Love numbers are quite insensitive to Earth structure, and tell little about the Earth's interior that is not much more accurately inferred from seismic body wave and free oscillation data. Even observations of the anomalous sidereal K_1 tide, which is affected by the core-mantle coupling, test more the mathematical theory of the coupling than the Earth model used in the theory. In these attempts to get accurate observations of the Love numbers, ocean load effects are an undesirable perturbation in the equilibrium body tide, which one attempts to remove by spatial averaging. Ocean loads, however, are world wide, and until the Earth tides have been measured many places on all the continents, there may always be a bias in these averaged Love numbers.

Much contemporary Earth tide research is turning from the study of the equilibrium body tide to the study of the load tide. Even here, most of the progress has merely been the demonstration that reasonable ocean tides placed on realistic Earth models produce load tides that do not wildly disagree with observation. In general, the configuration of the ocean tide is more poorly known than the load response of the Earth, so before the geophysical problem is tackled, it is necessary to have improved ocean tide models. Load tide studies can perhaps assist the oceanographers in this, because an acceptable ocean tide model must reproduce the geophysically observed load tides. A trivial constraint was determined by Farrell (1972*a*), who showed the ocean tide must conserve mass to explain several gravity tide observations. In terms of the spherical harmonic expansion of the tide, this means the coefficient of order 0 must vanish. A non-trivial constraint on the tides would be limits, determined geophysically, on the degree 2 coefficients of the tide. There are ten numbers to be determined, five each for the real and imaginary part of the tide, and from a particular two of them the dissipation of lunar tidal energy in the oceans can be found. This could best be done from a series of observations in the interiors of the continents, using the distance from the sea to filter out the load of small scale, local tides. On the other hand, a programme of coast-line and island observations is most useful for studying the tides in a particular ocean basin. Concurrent sea-floor recordings of the ocean tides would be valuable in interpreting the geophysical data.

In two unusual areas, the Bay of Fundy and the Irish Sea, the ocean tide is large and moderately well known. The tilt and gravity load tides are large near both regions (Beaumont & Lambert 1972; Farrell 1970). On the basis of tilt observations, Beaumont & Lambert (1972) have already been able to reject some proposed models of the local crustal structure. Their use of the finite element method in calculating the Green functions is attractive because it can easily handle laterally heterogeneous Earth models.

Much of this work has been done in close collaboration with Myrl Hendershott, who suggested the iterative procedure for solving the tide equations. The research has been supported by the National Science Foundation.

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